Theoretical Computer Science 134 (1994) 415–426
Elsevier

On the tiling of a torus with two bars

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Communicated by M. Nivat
Received December 1992
Revised October 1993

Abstract

Remila E, On the tiling of a torus with two bars, Theoretical Computer Science 134 (1994) 415–426.

We prove that, for two fixed integers m and n, the study of the tileability of a torus with h_m (the horizontal bar of length m and width 1) and v_n (the vertical bar of length n and width 1) can be limited to study of a finite number of cases.

1. Introduction

Tilings have always fascinated men. They are the subject of a lot of books, with many pictures [3]. Recently, the study of tilings has aroused a new interest, with algorithmic problems.

In particular, the tileability of a figure of the plane with two bars of unit width, a horizontal one of length m (tile h_m) and a vertical one of length n (tile v_n), m and n denoting integers such that $m \neq 1$ and $n \neq 1$, has been very studied. If $m > 2$ or $n > 2$, Robson [7] has proved that this problem is NP-complete in general. Nevertheless, linear algorithms have been constructed for restricted classes of figures: trapezes [6], figures "without bridge" [5], figures "without holes" [4]. Moreover, a very simple characterization of figures without holes admitting a unique tiling with h_m and v_n has been found [1]. Figure 1 is an example corresponding to $m = 6$ and $n = 3$.

Another subject of research is the study of tilings of a big rectangle with small rectangles of various sides. Some strong results have been obtained: First, de Bruijn

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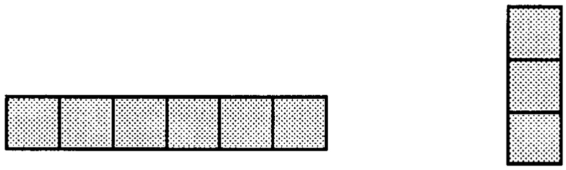

Fig. 1. Tiles h_6 and v_3.

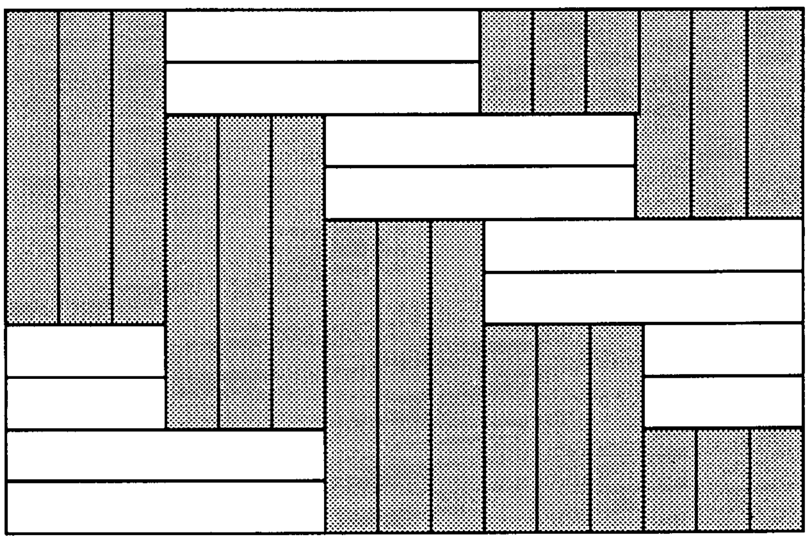

Fig. 2. Tiling of the torus $T_{15\times 10}$ with h_6 and v_6. By dividing the lengths and widths by 6, a torus with no integer side, tiled with rectangles with one integer side is constructed.

[2] proved that an $a \times b$ rectangle is tiled with copies of $c \times c$ or $d \times c$ rectangles, then integer c divides one of a, b. This result has been generalized in the following way: if each small rectangle has at least one integer side, then the big rectangle has at least one integer side [8]. Those results can be generalized when two opposite sides of the rectangle are identified in order to obtain a cylinder: We can say that tilings of a cylinder with rectangles have same properties as tilings of a rectangle with rectangles. All these results are lost when opposite sides of the rectangle are identified in order to obtain a torus: tilings of a torus with rectangles have different properties from tilings of a rectangle with rectangles. Robinson and Golomb discovered independently the example of Fig. 2. Moreover, Robinson found a necessary condition of a torus to be tiled, using groups generated by widths an lengths of tiles [8].

This paper studies the tileability of a torus with h_m and v_n. We prove that this problem can be simplified. After the simplification is done, if sides of the torus are large enough, then there exists a tiling of the torus. This proves that the previous study can be limited to a finite number of toruses.

We give an algorithm for tiling. In particular, this algorithm answers the following problem, due to Golomb: For which triples (a, b, k) can the torus $a \times b$ be tiled with h_k and v_k? We explicitly give solutions for $k=6$, which is the lowest integer for which the above question is not trivial.

2. Definitions

The *torus* $T_{a\times b}$ (where a and b denote nonnegative integers) is the product space $(\mathbb{Z}/a\mathbb{Z})\times(\mathbb{Z}/b\mathbb{Z})$.

We say that each element (x, y) of the torus $T_{a\times b}$ is a *cell.*

Each cell has four *neighbors*: Cells $(x-1, y)$, $(x+1, y)$, $x, y+1)$ and $(x, y-1)$ are the *left*, *right*, *upper* and *lower neighbors* of cell (x, y), respectively.

A *figure* F is a finite set of cells. Let (i,j) be an ordered pair of integers. The set of cells $(x+i,\ y+j)$, where (x, y) is a cell of F, is denoted by $F+(i,j)$.

A cell (x, y) is on the *left* (resp., *right*, *upper* or *lower*) *side* of a figure F if (x, y) is an element of F and the left (resp., right, upper or lower) neighbor of (x,y) is not an element of F.

A figure R is called a $c\times d$ *rectangle* (c and d denoting integers such that $0<c\leqslant a$ and $0<d\leqslant b$) if it satisfies the following property: There exists a cell (x, y) such that F is the set of the cells (ξ, η) such that we have $x\leqslant\xi<x+c$ and $y\leqslant\eta<y+d$. We say that c is the *length* of R and d is the *height* of R.

For $c\neq a$ and $d\neq b$, the cells (x, y), $(x+c-1, y)$, $(x, y+d-1)$ and $(x+c-1, y+d-1)$ are called the *lower left*, *lower right*, *upper left* and *upper right corners* of the rectangle, respectively.

Let m and n be fixed nonnegative integers. A h_m *tile* is an $m\times 1$ rectangle and a v_n *tile* is a $1\times n$ rectangle. A tiling $\boldsymbol{P}$ of a figure F with the h_m and v_n tiles is a set of disjoint tiles whose union is figure F.

A *horizontal block* of a figure F is an $m\times j$ rectangle, with $j<n$, included in F. A *vertical block* of a figure F is an $i\times n$ rectangle, with $i<m$, included in F. Clearly, each block admits a tiling with h_m and v_n, which only uses one kind of tile.

Let $\boldsymbol{P}$ be a tiling of the torus $T_{a\times b}$. We define a mapping f_P from the set of cells of $T_{a\times b}$ to the set $\{h_0, h_1, \ldots, h_{m-1}, v_0, v_1, \ldots, v_{n-1}\}$ by

(i) $f_P((x, y))=h_i$ if cell (x, y) is an element of a h_m tile H of $\boldsymbol{P}$ and cell $(x-i, y)$ is the left cell of H,

(ii) $f_P((x, y))=v_j$ if cell (x, y) is an element of a v_n tile V of $\boldsymbol{P}$ and cell $(x, y-j)$ is the lower cell of V.

Remark 2.1. Clearly, we have $f_P=f_{P'} \Rightarrow \boldsymbol{P}=\boldsymbol{P}'$.

Remark 2.2. Let f be a mapping from the set of cells of $T_{a\times b}$ to the set $\{h_0, h_1, \ldots, h_{m-1}, v_0, v_1, \ldots, v_{n-1}\}$ satisfying the following properties:

(a) for $i>0, f((x, y))=h_i \Rightarrow f((x-1, y))=h_{i-1}$,

(b) for $i<m-1, f((x,y))=h_i \Rightarrow f((x+1,y))=h_{i+1}$,
(c) for $j>0, f((x,y))=v_j \Rightarrow f((x,y-1))=v_{j-1}$,
(d) for $i<n-1, f((x,y))=v_j \Rightarrow f((x,y+1))=v_{j+1}$.
Then, there exists a unique tiling $\boldsymbol{P}$ such that $f=f_{\boldsymbol{P}}$.

3. Reduction of the problem

Proposition 3.1. *Let a, b, m, n and k be five nonnegative integers. The torus $T_{a\times b}$ is tileable with h_m and v_n if and only if the torus $T_{ka\times b}$ is tileable with h_{km} and v_n.*

Proof. At first, assume that the torus $T_{a\times b}$ is tileable with h_m and v_n. Let $\boldsymbol{P}$ be a tiling of the torus $T_{a\times b}$ with h_m and v_n. A tiling $\boldsymbol{kP}$ of the torus $T_{ka\times b}$ with h_{km} and v_n is constructed in the following way: Let (x,y) be a cell of the torus $T_{a\times b}$.

(i) If $f_{\boldsymbol{P}}((x,y))=h_i$, then for each integer k' such that $0\leqslant k'<k$ we have $f_{\boldsymbol{kP}}((kx+k',y))=h_{ki+k'}$.

(ii) If $f_{\boldsymbol{P}}((x,y))=v_j$, then for each integer k' such that $0\leqslant k'<k$ we have $f_{\boldsymbol{kP}}((kx+k',y))=v_j$.

Remark that function $f_{\boldsymbol{kP}}$ satisfies the hypothesis of Remark 2.2, which proves the existence of $\boldsymbol{kP}$.

Informally, we proceed as follows: From the torus $T_{a\times b}$, we do a horizontal dilatation of factor k. Thus, we obtain a tiling of the torus $T_{ka\times b}$ with h_{km} tiles and $k\times n$ rectangles. Afterwards, we divide each $k\times n$ rectangle into v_n tiles.

The converse part of the proof is based on a idea of Robson [7] about a similar problem. Assume that the torus $T_{ka\times b}$ is tileable with h_{km} and v_n. Let $\boldsymbol{P}'$ be a tiling of the torus $T_{ka\times b}$ with h_{km} and v_n. A tiling $\boldsymbol{k}^{-1}\boldsymbol{P}'$ of the torus $T_{a\times b}$ with h_m and v_n is constructed on the following way: Consider, (kx,y) be a cell of the torus $T_{ka\times b}$.

(i) If $f_{\boldsymbol{P}'}((kx,y))=h_{kq+r}$, with $0\leqslant r<k$, then we have $f_{k^{-1}\boldsymbol{P}}((x,y))=h_q$,

(ii) If $f_{\boldsymbol{P}'}((kx,y))=v_j$, , then we have $f_{k^{-1}\boldsymbol{P}}((x,y))=v_j$,.

Remark that function $f_{k^{-1}\boldsymbol{P}'}$ satisfies the hypothesis of Remark 2.2, which proves the existence of $\boldsymbol{k}^{-1}\boldsymbol{P}$.

Informally, we have only taken columns of the torus $T_{ka\times b}$ whose number is a multiple of integer k (see Fig. 3). □

Proposition 3.1 permits to study the tileability of the torus $T_{a\times b}$ with h_m and v_n only when we have $\gcd(a,m)=\gcd(b,n)=1$. Moreover, we have the following remark:

Remark 3.2. Let a, b, m and n be four nonnegative integers such that we have $\gcd(a,m)=\gcd(b,n)=1$. If the torus $T_{a\times b}$ is tileable with h_m and v_n, then we have $\gcd(m,n)=1$.

The above result is proved as follows: If m and n were both multiple of a primary integer p, then, by looking at the numbers of cells of the torus $T_{a\times b}$, we obtain that

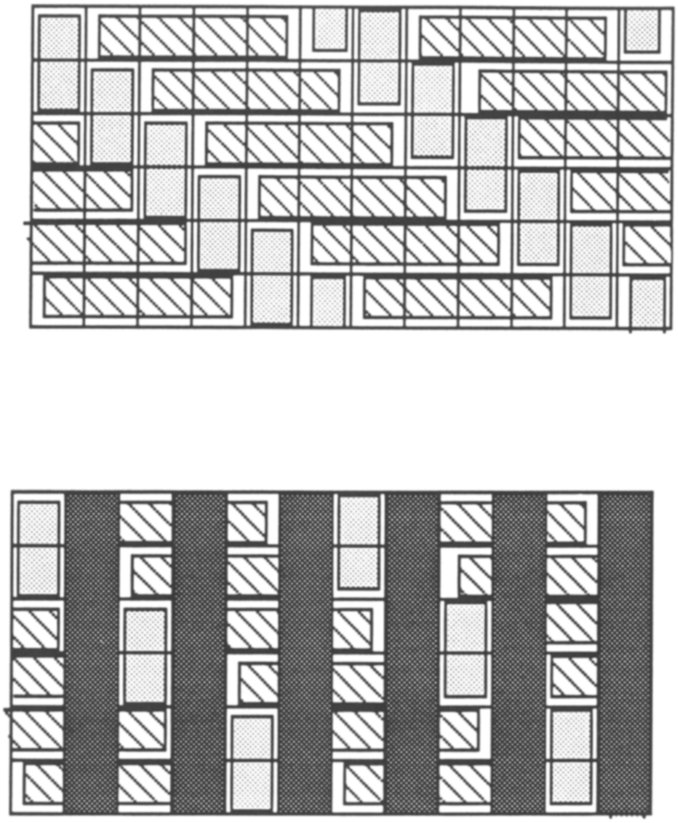

Fig. 3. Method to prove converse part of Proposition 3.1.

product ab is a multiple of p, which means that either a or b, is a multiple of p. This contradicts the hypothesis $\gcd(a, m) = \gcd(b, n) = 1$. $\square$

To conclude this part, we may study the tileability of the torus $T_{a \times b}$ with h_m and v_n only when we have $\gcd(a, m) = \gcd(b, n) = \gcd(m, n) = 1$.

4. Case when both a and b are large

In this part, we assume that we have $\gcd(a, m) = \gcd(b, n) = \gcd(m, n) = 1$. Informally, we prove that if a and b are both sufficiently large, then the torus $T_{a \times b}$ is tileable with h_m and v_n. Formally, we have the following proposition.

Proposition 4.1. *Let a, b, m and n be nonnegative integers such that we have* $\gcd(a, m) = \gcd(b, n) = \gcd(m, n) = 1$. *Let d, b' and c be integers such that*

(a) *$a \equiv nd[m]$ and $0 \leqslant d < m$ (remark that d exists, since we have* $\gcd(m, n) = 1$),

(b) $b \equiv b'$ $[n]$ *and* $0 \leqslant b' < n$,

(c) $b'd + ca \equiv 0$ $[m]$ *and* $0 \leqslant c < m$ *(remark that c exists, since we have* $\gcd(a, m) = 1$*).*
Let α *and* β *be integers defined by* $((c+1)n + b')d = \alpha$ *and* $(c+1)n + b' = \beta$. *Assume that we have* $a \geqslant \alpha$ *and* $b \geqslant \beta$. *Then the torus* $T_{a \times b}$ *is tileable with* h_m *and* v_n.

Proof. The proof of this proposition involves the following two lemmas.

Lemma 4.2. *Let* (i, j) *be an ordered pair of positive integers. Assume that the torus* $T_{\alpha \times \beta}$ *is tileable with* h_m *and* v_n. *Then the torus* $T_{(\alpha + mi) \times (\beta + nj)}$ *is tileable with* h_m *and* v_n.

Proof. Let $\boldsymbol{P}$ be a tiling of the torus $T_{\alpha \times \beta}$ with h_m and v_n. For $0 \leqslant x < \alpha$, we define integer c_x in the following way:

(a) if we have $f_{\boldsymbol{P}}((x, \beta - 1)) = v_p$, then $c_x = p$,

(b) if we have $f_{\boldsymbol{P}}((x, \beta - 1)) = h_p$, then $c_x = n - 1$.

Similarly, for $0 \leqslant y < b$ we define integer c_x in the following way:

(a) if we have $f_{\boldsymbol{P}}((\alpha - 1, y)) = h_p$, then $c_y = p$,

(b) if we have $f_{\boldsymbol{P}}((\alpha - 1, y)) = v_p$, then $c_y = m - 1$.

A tiling $\boldsymbol{P}'$ of the torus $T_{(\alpha + mi) \times (\beta + nj)}$ with h_m and v_n is constructed in the following way: Let (x, y) be a cell of the torus $T_{(\alpha + mi) \times (\beta + nj)}$.

(i) If we have $0 \leqslant x < \alpha$ and $0 \leqslant y < \beta$, then we have $f_{\boldsymbol{P}'}((x, y)) = f_{\boldsymbol{P}}((x, y))$.

(ii) If we have $0 \leqslant x < \alpha + mi$ and $\beta \leqslant y < \beta + nj$, then we have $f_{\boldsymbol{P}'}((x, y)) = h_r$, where r denotes the integer such that $x - \alpha \equiv r[m]$ and $0 \leqslant r < m$.

(iii) If we have $\alpha \leqslant x < \alpha + mi$ and $0 \leqslant y < b$, then we have $f_{\boldsymbol{P}'}((x, y)) = h_r$, where r denotes integer such that $x - \alpha + 1 + c_y \equiv r[m]$ and $0 \leqslant r < m$.

(iv) If we have $0 \leqslant x < \alpha$ and $\beta \leqslant y < \beta + nj$, then we have $f_{\boldsymbol{P}'}((x, y)) = v_r$, where r denotes the integer such that $y - \beta + 1 + c_y \equiv r[n]$ and $0 \leqslant r < m$.

Remark that function $f_{\boldsymbol{P}'}$ satisfies the hypothesis of Remark 2.2, which proves the existence of $\boldsymbol{P}'$.

Informally (see Fig. 4), cells (x, y) of case (i) are tiled with tiles of $\boldsymbol{P}$. Cells (x, y) of case (iii) are tiled with h_m tiles, whose left or right part possibly covers some cells of case (i). Similarly, cells (x, y) of case (iv) are tiled with v_n tiles, whose upper or lower part possibly covers some cells of case (i). Cells (x, y) of case (ii) form an $mi \times nj$ rectangle, which can be tiled by using only h_m tiles. $\square$

Lemma 4.3. *Let a, b, m and n be nonnegative integers such that we have* $\gcd(a, m) = \gcd(b, n) = \gcd(m, n) = 1$. *Let* α, β *be defined as in Lemma* 4.2. *Then the torus* $T_{\alpha \times \beta}$ *is tileable with* h_m *and* v_n.

Proof. Let V_0 be the $d \times n$ rectangle whose lower left corner is cell $(0, 0)$. For each integer i, such that $0 \leqslant i < \beta$, let V_i be the rectangle defined by the equality $V_i = V_0 + (id, i)$.

Let (x, y) be a cell of the torus $T_{\alpha \times \beta}$ such that $0 \leqslant x < \alpha$. Remark that if (x, y) is an element of V_i, then we have $id \leqslant x < (i+1)d$. Thus, blocks V_i are disjoint. Let V be the

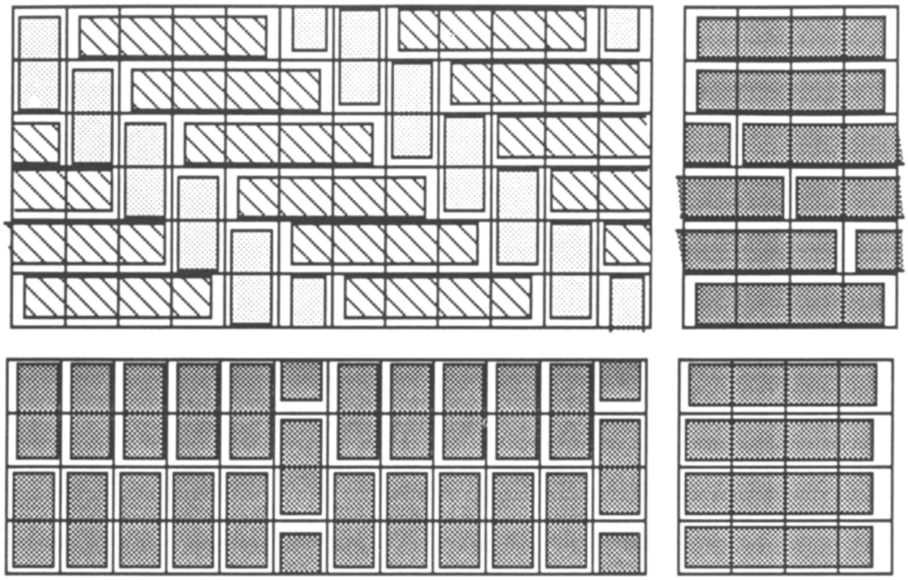

Fig. 4. Proof of Lemma 4.2.

figure of $T_{\alpha\times\beta}$, defined by $V=V_0\cup V_1\cdots\cup V_{\beta-1}$. Notice that V is tileable with h_m and v_n by using only v_n tiles.

Let p be an integer such that $0\leqslant p<\beta-1$, and let L_p be the set of cells (x,p) of the torus $T_{\alpha\times\beta}$ (informally, L_p is a line of the torus $T_{\alpha\times\beta}$). Notice that the sets L_p are disjoint.

(a) For $n-1\leqslant p<\beta$, we have $L_p\cap V=L_p\cap(V_p\cup V_{p-1}\cdots\cup V_{p-n+1})$, since for each integer i, such that $0\leqslant i<\beta$, the cells of V_i are cells (x,y) with $i\leqslant y<i+n$. Thus, we have $l_p\cap V=(L_p\cap V_p)\cup(L_p\cap V_{p-1})\cdots\cup(L_p\cap V_{p-n+1})$, which proves that $L_p\cap V$ is the $nd\times 1$ rectangle formed from cells (x,p) such that we have $(p-n+1)d\leqslant x<(p+1)d$. Hence, figure L_p-V is a $(cn+b')d\times 1$ rectangle (since $(cn+b')d+nd=\alpha$). Remark that $(cn+b')d\equiv 0\,[m]$, from the definition of integers c and d. Thus, rectangle L_p-V is tileable with h_m and v_n by using only h_m tiles.

(b) Similarly, for $0\leqslant p\leqslant n-1$, we have $L_p\cap V=L_p\cap(V_p\cup V_{p-1}\cdots\cup V_0\cup V_{\beta-1}\cup V_{\beta-2}\cdots\cup V_{\beta-n+p+1})$, which proves that $L_p\cap V$ is the $nd\times 1$ rectangle formed from cells (x,p) such that $0\leqslant x<(p+1)d$ or $(\beta-n+p+1)d\leqslant x<\beta d$ ($L_p\cap V$ is a rectangle since cell $(\beta d-1,p)$ is the left neighbor of cell $(0,p)$). Hence, L_p-V is a $(cn+b')d\times 1$ rectangle, which is tileable with h_m and v_n by using only h_m tiles.

Now, remark that the union of tilings of V_i and V_p gives a tiling $\boldsymbol{P}$ of the torus $T_{\alpha\times\beta}$. □

Proof of Proposition 4.1 (*conclusion*). We notice that $a-\alpha$ is a multiple of m and $b-\beta$ is a multiple of n, from the definition of integers d, b' and c. □

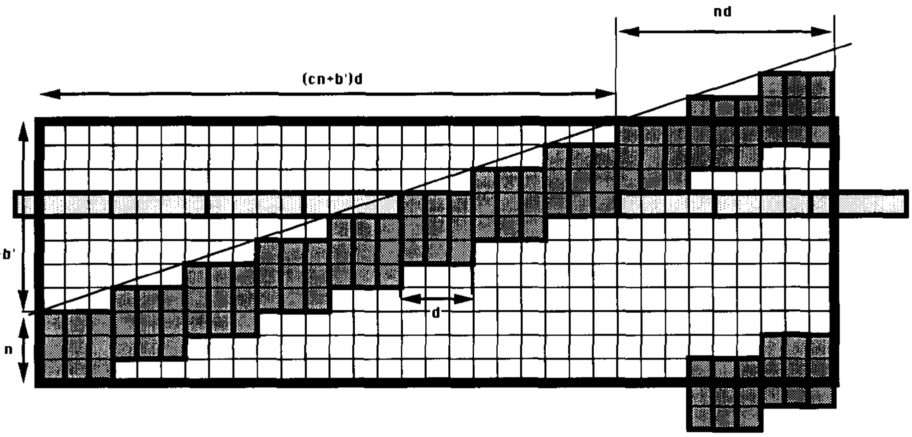

Fig. 5. Tiling of the torus $T_{\alpha \times \beta}$.

Now, we see what happens when one of integers a and b is not large. That is the aim of Section 5.

5. Case when b is a fixed integer

Proposition 5.1. *Let b, a', m and n be nonnegative integers such that $0 \leqslant a' < m$. Assume that there exists a positive integer k such that the torus $T_{(mk+r)\times b}$ is tileable with h_m and v_n. Then, there exists a positive integer k' such that*

(i) *the torus $T_{(mk'+a')\times b}$ is tileable with h_m and v_n,*

(ii) $0 \leqslant k' < (m+1)^b$.

Proof (*cf. Fig. 6*). Let k'' be the lowest positive integer such that the torus $T_{(mk''+a')\times b}$ is tileable with h_m and v_n. Assume that we have $k'' \geqslant (m+1)^b$. Let P be a tiling of the torus $T_{(mk''+a')\times b}$. Let k be an integer such that $0 \leqslant \kappa \leqslant k''$. We define string $s(\kappa) = w_0 w_1 \cdots w_{b-1}$, on alphabet $\{h_0, h_1, \ldots, h_{m-1}, v\}$, in the following way:

(a) if $f_P\ ((m\kappa, i)) = h_j$, when $w_i = h_j$,

(b) if $f_P\ ((m\kappa, i)) = v_j$, when $w_i = v$.

Remark that $\{h_0, h_1, \ldots, h_{m-1}, v\}^*$ contains $(m+1)^b$ strings of length b. Thus, there exists a pair (k_1, k_2) of integers such that $k_1 < k_2$ and $s(k_1) = s(k_2)$ (informally, that means that rows whose numbers are mk_1 and mk_2 are tiled by the same way). A tiling P' of the torus $T_{(m(k''-k_2+k_1)+a')\times b}$ is obtained as follows:

(a) for each cell (x, y) such that $0 \leqslant x \leqslant mk_1$, we have

$$f_{P'}((x, y)) = f_P((x, y)),$$

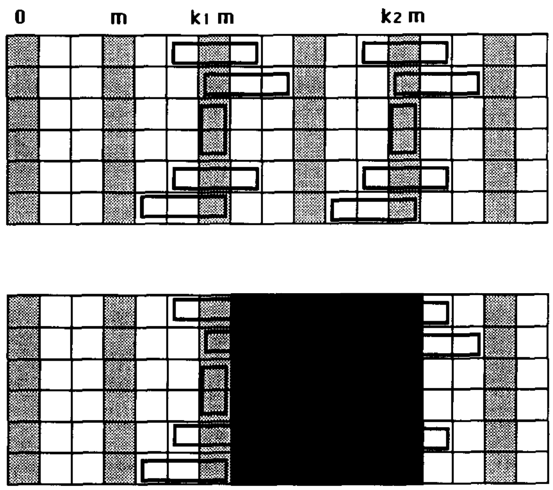

Fig. 6. Proof of Proposition 5.1.

(b) for each cell (x, y) such that $mk_1 < x < m(k'' - k_2 + k_1) + a'$, we have

$$f_{P'}((x, y)) = f_P((x + (m(k_2 - k_1), y)).$$

Remark that function $f_{P'}$ satisfies the hypothesis of Remark 2.2, which proves the existence of $\boldsymbol{P}'$ (informally, tiling $\boldsymbol{P}'$ is obtained by taking the torus $T_{(mk''+a')\times b}$, tiled by $\boldsymbol{P}$, and deleting cells (x, y) such that $mk_1 < y \leqslant mk_2$).

The existence of $\boldsymbol{P}'$ gives a contradiction. □

By using Lemma 4.2, we have the following alternatives:

(i) there exists a positive integer k, with $0 \leqslant k < (m+1)^b$, such that the torus $T_{(mk+a')\times b}$ is tileable with h_m and v_n. In this case, for each positive integer k' such that $k < k'$, the torus $T_{(mk'+a')\times b}$ is tileable with h_m and v_n.

(ii) there exists no positive integer k, such that $0 \leqslant k < (m+1)^b$ and the torus $T_{(mk+a')\times b}$ is tileable with h_m and v_n. In this case, for each positive integer k', the torus $T_{(mk'+a')\times b}$ is not tileable with h_m and v_n.

6. Description of the algorithm

The algorithm to know if the torus $T_{a\times b}$ is tileable with h_m and v_n proceeds as follows:

Compute integers m', n', x and y such that
$\gcd(a, m)m' = m$, $\gcd(a, m)x = a$, $\gcd(b, n)n' = n$ and $\gcd(b, n)y = b$.
If $\gcd(m', n') \neq 1$, then the torus $T_{a\times b}$ is not tileable with h_m and v_n.

Else,
do $m := m'$, $n := n'$, $a := x$ and $b := y$,
compute integers a', b', c, α and β such that
(i) $a \equiv a'[m]$ and $0 \leqslant a' < m$,
(ii) $b = b'[n]$ and $0 \leqslant b' < n$,
(iii) $a \equiv nd[m]$ and $0 \leqslant d < m$,
(iv) $b'd + ca \equiv 0\ [m]$ and $0 \leqslant c < m$,
(v) $((c+1)n + b')d = \alpha$,
(vi) $(c+1)n + b' = \beta$.
If $a \geqslant \alpha$ and $b \geqslant \beta$, then the torus $T_{a \times b}$ is tileable with h_m and v_n.
Else,
if $b < \beta$, then
compute integer k such that $a = mk + a'$.
If there exists an integer k' such that the torus $T_{(mk'+a') \times b}$ is tileable with h_m and v_n and $0 \leqslant k' < \min(k, (m+1)^b)$, then the torus $T_{a \times b}$ is tileable with h_m and v_n,
Else, the torus $T_{a \times b}$ is not tileable with h_m and v_n,
Else
compute integer k such that $b = nk + b'$.
If there exists an integer k' such that the torus $T_{a \times (nk'+b')}$ is tileable with h_m and v_n and $0 \leqslant k' < \min(k, (n+1)^a)$, then the torus $T_{a \times b}$ is tileable with h_m and v_n,
Else, the torus $T_{a \times b}$ is not tileable with h_m and v_n.

Remark 6.1. There exists a finite number of ordered pairs (m', n') such that m' divides m and n' divides n. In the last part of this algorithm, for each pair (m', n'), we have a finite set of toruses for which we need to know if they are tileable with $h_{m'}$ and $v_{n'}$ or not. From proposition 3.1, that means that, to use the above algorithm, there is a finite set X of toruses for which we need to know if they are tileable with h_m and v_n or not. Set X is clearly known, thus the above condition can be realized in a finite time. Moreover, we can get a tiling for each element of X in a finite time.

Remark 6.2. When the torus $T_{(mk+a') \times (nk'+b')}$ is tileable with h_m and v_n, the algorithm gives an ordered pair of integers (r, r') such that $r \leqslant k$, $r' \leqslant k'$, and the torus $T_{(mr+a') \times (nr'+b')}$ is tileable with h_m and v_n. The torus $T_{(mr+a') \times (nr'+b')}$ is an element of X or the torus $T_{(mr+a') \times (nr'+b')}$ is equal to the torus $T_{\alpha \times \beta}$ of Section 4. Thus, a tiling of the torus $T_{(mk+a') \times (nk'+b')}$ can easily be constructed from Lemma 4.2.

7. Special cases

7.1. $m = p^k$ and $n = p^{k'}$, with p prime

We have the following result.

Proposition 7.1. *Let m and n be nonnegative integers. The class* $\mathbf{C}$ *of the toruses $T_{a\times b}$ tileable with h_m* and *v_n is equal to the class $\mathbf{C}'$ of the toruses $T_{a\times b}$ such that a is a multiple of m or b is a multiple of n if and only if there exists k and k' two integers, and p a prime number such that $m=p^k$ and $n=p^{k'}$.*

Proof. This is a direct consequence of Proposition 3.1. □

7.2. $m=n$:

Golomb has put the following question: For which triples (a,b,k) can the torus $T_{a\times b}$ be tiled with h_k and v_k? The above algorithm gives an interesting answer. We have exhaustively studied the case where $k=6$, which is the lowest integer for which the problem is not trivial.

Proposition 7.2. *The torus $T_{a\times b}$ is tileable with h_6 and v_6 if and only if it satisfies at least one of the following conditions:*

(i) *a is a multiple of 6 or b is a multiple of 6,*
(ii) *a is even, b is a multiple of 3, $a\geqslant 10$, $b\geqslant 15$ and $(a,b)\neq(14,15)$,*
(iii) *b is even, a is a multiple of 3, $b\geqslant 10$, $a\geqslant 15$ and $(b,a)\neq(14,15)$.*

Proof. We only study what happens when a is even and b is a multiple of 3. In this case, we study the tileability of the torus $T_{a'\times b'}$ with h_3 and v_2, where a' and b' are defined by $2a'=a$ and $3b'=b$. Remark that the torus $T_{5\times 5}$, the torus $T_{10\times 5}$ and the torus $T_{7\times 7}$ are tileable with h_3 and v_2 (see Fig. 7). Thus, from Lemma 4.2, if $a'\geqslant 5$, $b'\geqslant 5$ and $(a',b')\neq(7,5)$, then the torus $T_{a'\times b'}$ is tileable with h_3 and v_2.

Moreover, if $a'\leqslant 3$ or $b'\leqslant 2$ or $b'=4$, the study of the tileability of the torus $T_{a'\times b'}$ with h_3 and v_2 is obvious. Hence, we have to study the following cases:

(1) $a'=4$, (2) $b'=3$, (3) $(a',b')=(7,5)$.

To do it, we use the following result.

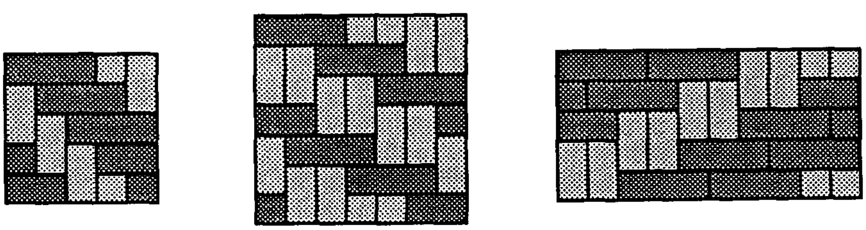

Fig. 7. Tilings of the torus $T_{5\times 5}$, the torus $T_{10\times 5}$ and the torus $T_{7\times 7}$ with h_3 and v_2.

Proposition 7.3. *The cylinder $C_{a\times b}$ (i.e. the product space $\{1,2,\dots,a\}\times\mathbb{Z}/b\mathbb{Z}$) is tileable with h_m* and *v_n if and only if a is a multiple of m or b is a multiple of n.*

Proof. This proposition is a direct consequence of a proposition (see [8]) which claims that whenever a cylinder is tiled with rectangles each of which has at least one integer side, then the cylinder has at least one integer side. □

Proof of Proposition 7.2 (*conclusion*). It is very easy to see that, for cases (1) and (2) if the torus $T_{a'\times b'}$ is tileable with h_3 and v_2, then cylinder $C_{a'\times b'}$ is tileable with h_3 and v_2. For case (3), we see that if the torus $T_{7\times 5}$ were tileable with h_3 and v_2, then cylinder $C_{7\times 5}$ would be tileable with h_3 and v_2, which is impossible. □

8. Open problems

Tilings of the torus are still an unclear subject. Informally, we give the following problems, which seem very near from what has been done in this paper.

(1) *Unicity*: For which quadruples (a,b,m,n) does the torus $T_{a\times b}$ have a unique tiling with h_m and v_n (up to translation and horizontal and vertical symmetries)? Find a polynomial algorithm which answers this question.

(2) *Torus of higher dimensions*: For example, given a 3-dimensional torus $T_{a\times b\times c}$, and a set $\{x_m, y_n, z_p\}$ of three orthogonal parallelopipeds whose sections are 1×1 squares, can the torus $T_{a\times b\times c}$ be tiled with x_m, y_n or z_p? Find a polynomial algorithm which answers this question.

(3) *Tiling of a torus with two rectangles*: Let R_1 and R_2 be rectangles of the torus $T_{a\times b}$. Can the torus $T_{a\times b}$ be tiled with copies of R_1 and R_2? Find a polynomial algorithm which answers this question.

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